**The Chinese University of Hong Kong**
**Department of Mathematics**
**MMAT 5340 Homework 7**
**Please submit your assignment online on Blackboard**
**Due at 18:00 p.m. on Monday, 24th Mar, 2025**

1. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$\begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.7 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0.9 & 0 \end{bmatrix}$$

   (a) Which states are transient and which are recurrent?

   (b) Is this markov chain irreducible or reducible?

   **Solution:**

   (a) Note that

$$P(1, 2) = 0.4, \quad P(2, 1) = 3$$

   then state 1 and state 2 are intercommunicate, i.e. $1 \leftrightarrow 2$. Also note that $P(3, 1) = 0.5$ and

$$\begin{aligned} P^2(1, 3) &= P(X_2 = 3|X_1 = 2)P(X_1 = 2|X_0 = 1) + P(X_2 = 3|X_1 = 4)P(X_1 = 4|X_0 = 1) \\ &= P(2, 3)P(1, 2) + P(4, 3)P(1, 4) = 0.7 \times 0.4 + 0.9 \times 0.4 \\ &= 0.64 \end{aligned}$$

   Hence $1 \leftrightarrow 3$ and furthermore $2 \leftrightarrow 3$. Similarly, we can show $P(1, 4) = 0.4$ and

$$P^2(4, 1) = P(4, 3)P(3, 1) + P(4, 2)P(2, 1) = 0.48$$

   then $1 \leftrightarrow 4$ and furthermore $2 \leftrightarrow 4$, $3 \leftrightarrow 4$.
   Applying following theorem
   " If a Markov chain has finite state space, there is at least one recurrent state" and the fact that all states in $S$ are intercommunicate, we conclude that all state are recurrent.

   (b) Since all states are intercommunicate then this Markov chain is irreducible.

2. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with state space $S = \{1, 2, 3\}$ and transition matrix

$$\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.4 & 0 & 0.6 \end{bmatrix}$$

   (a) Which states are transient and which are recurrent?

(b) Is this markov chain irreducible or reducible?

**Solution:**

(a) Note that

$$P_3(\tau_3 \geq n) = P(X_1 = X_2 = \cdots = X_{n-1} = 1 | X_0 = 3) = 0.4(0.5)^{n-2}.$$

Letting $n \to \infty$ on both sides of above formula yields

$$P_3(\tau_3 = \infty) = 0$$

then

$$P_3(\tau_3 < \infty) = 1 - P_3(\tau_3 = \infty) = 1$$

Hence state 3 is recurrent. Since $P_{1,3} = 0.5 \neq 0$ and $P_{3,1} = 0.4 \neq 0$, then state 1 and state 3 are intercommunicate. Hence state 1 also recurrent.
We also note that $P^n(2,2) = 1, n = 1, 2, 3, \cdots$
then

$$\sum_{n=1}^{\infty} P^n(2,2) = \sum_{n=1}^{\infty} 1 = \infty.$$

Hence state 2 is recurrent .

(b) Since $1 \nrightarrow 2$ then state 1 and state 2 are not intercommunicate. Also $2 \nrightarrow 3$ implies that state 2 and state 3 are not intercommunicate. Hence this Markov chain is reducible.